

Clairaut's Theorem

10/4/21

If $f(x,y)$ has cts. second-order partial derivatives on an open disk D , then $\frac{d^2f}{dx dy} = \frac{d^2f}{dy dx}$ on D .

Notation: $f_x = \frac{df}{dx}$, $f_y = \frac{df}{dy}$

$$f_{xx} = \frac{d^2f}{(dx)^2}$$

$$f_{xy} = (f_x)_y = \frac{d}{dy} \left(\frac{d}{dx} (f) \right) = \frac{d^2f}{dy dx}$$

Pf: Let $f(x,y)$ has cts. second-order mixed partial derivatives on some open disk D and suppose $(a,b) \in D$.

$$\text{Let } \Delta(h) := (f(a+h, b+h) - f(a+h, b)) - (f(a, b+h) - f(a, b))$$

for all $h \neq 0$ where $(a+h, b+h), (a+h, b), (a, b+h) \in D$

Let $\alpha(x) := f(x, b+h) - f(x, b)$ and notice $\Delta(h) = \alpha(a+h) - \alpha(a)$

For h fixed, we can apply the MVT to obtain c_h satisfying $|a - c_h| \leq |h|$ and $\alpha'(c_h)h = \alpha(a+h) - \alpha(a)$. Thus

$$\Delta(h) = \alpha(a+h) - \alpha(a) = h \alpha'(c_h) = h(f_x(c_h, b+h) - f_x(c_h, b))$$

Let $B(y) := f_x(c_h, y)$, we see again by MVT there is d_h satisfying $|b - d_h| \leq |h|$ and $B'(d_h)h = f_x(c_h, b+h) - f_x(c_h, b)$

$$\text{Thus } \Delta(h) = h(f_x(c_h, b+h) - f_x(c_h, b)) = h(h B'(d_h)) = \underbrace{h^2 f_{xy}(c_h, d_h)}_{(*)}$$

If we rearrange $\Delta(h) = (f(a+h, b+h) - f(a, b+h)) - (f(a+h, b) - f(a, b))$,
 we can repeat the argument (using y first) to obtain δ_h, δ_h
 satisfying $|a - \gamma_h| \leq |h|$, $|b - \delta_h| \leq |h|$ and
 $\Delta(h) = h^2 f_{xy}(\gamma_h, \delta_h)$ for all fixed h .

(*)

Notice $\lim_{h \rightarrow 0} (c_h, d_h) = (a, b) = \lim_{h \rightarrow 0} (\gamma_h, \delta_h)$ (by construction)

Thus we compute:

$$f_{xy}(a, b) = f_{xy}(\lim_{h \rightarrow 0} (c_h, d_h))$$

$$\text{By continuity} \rightarrow \lim_{h \rightarrow 0} f_{xy}(c_h, d_h)$$

$$\text{by } (*) = \lim_{h \rightarrow 0} \frac{\Delta(h)}{h^2}$$

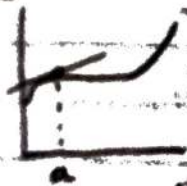
$$\text{by } (*) = \lim_{h \rightarrow 0} f_{xy}(\gamma_h, \delta_h)$$

$$= f_{xy}(\lim_{h \rightarrow 0} (\gamma_h, \delta_h))$$

$$= f_{xy}(a, b). \quad \square$$

S14.1: Linear Approximation of Multivariable Functions

IDEA: In Calc I, we say the tangent line to f at a "well-approximates"
 f near $(a, f(a))$.



as $x \rightarrow a$, the error approximating
 f w/ the tangent line goes to 0.

In Calc III we use a tangent ^(hyper) plane instead (nice to be approximating)

Small changes in input have change for output of f measured by the first derivatives. Calc I - $f(x) \approx y = f(a) + f'(a)(x-a)$ near input a

In Calc III, these changes are measured by:

$$f(x,y) \approx z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad \text{near } (a,b)$$

Ex. Find an equation of the tangent plane to $f(x,y) = x^2 + xy - y^2$ at $(4,1)$

Sol. The tangent plane has equation $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$


$$f(4,1) = 4^2 + 4 \cdot 1 - 1^2 = 19$$

$$f_x(x,y) = 2x + y$$

$$f_x(4,1) = 2 \cdot 4 + 1 = 9$$

$$f_y(x,y) = x - 2y$$

$$f_y(4,1) = 4 - 2 = 2$$

Therefore the desired plane is $z = 19 + 9(x-4) + 2(y-1)$ 

Ex. Find tangent plane to $f(x,y) = \frac{e^{x-y}}{x}$ at $(2,2,\frac{1}{2})$.

Sol. $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$


$$f(2,2) = \frac{1}{2}$$

$$f_x(x,y) = \frac{x e^{x-y} - e^{x-y}}{x^2}$$

$$f_x(2,2) = \frac{1}{4}$$

$$f_y(x,y) = -\frac{e^{x-y}}{x}$$

$$f_y(2,2) = -\frac{1}{2}$$

$$z = \frac{1}{2} + \frac{1}{4}(x-2) - \frac{1}{2}(y-2)$$
 

In Calc I, we also thought from the perspective of differentials
 $\Delta f \approx f'(a) \Delta x$ at $a=x$

\uparrow
 changing f
 from change in x

\uparrow
 change
 in x

For functions with 2 variables

$$\Delta f \approx f_x(a,b) \Delta x + f_y(a,b) \Delta y$$

In calc I, Δ replaced by symbols and inserted equalities

$$df = f'(x) dx \quad \text{ie} \quad df = \frac{df}{dx} dx$$

Defn: The total differential of function f of variable x_1 to x_n is

$$df = \frac{df}{dx_1} dx_1 + \frac{df}{dx_2} dx_2 + \dots + \frac{df}{dx_n} dx_n$$

→ represents change in x

represent loose changes in x

Ex. Compute total differential of $F(x,y,z) = \frac{1}{2} \frac{\ln(x-3y)}{z}$

Sol. Compute

$$f_x(x,y,z) = \frac{1}{2} \cdot \frac{1}{x-3y} = \frac{1}{2(x-3y)} \quad f_y(x,y,z) = \frac{-3}{(x-3y)z}$$

$$f_z(x,y,z) = -\frac{\ln(x-3y)}{z^2}$$

$$df = f_x dx + f_y dy + f_z dz = \frac{1}{2(x-3y)} dx - \frac{3}{2(x-3y)z} dy - \frac{\ln(x-3y)}{z^2} dz$$

Ex. Estimate change in F from $(4, 1, 1)$ to $(4.5, 1.5, .5)$

$$\text{Sol. } \Delta f \approx df = \frac{1}{2(x-3y)} dx - \frac{3}{2(x-3y)z} dy - \frac{\ln(x-3y)}{z^2} dz$$

$$\Delta f \approx f_x(4,1,1) \Delta x + f_y(4,1,1) \Delta y + f_z(4,1,1) \Delta z$$

$$= \frac{1}{2(4.5-4)} + 3(1.5-1) - \frac{\ln(1)}{1} (.5-1)$$

$$= \frac{1}{2} - \frac{3}{2} - 0 = -1$$

